The Existence and Uniqueness Theorem for a Cauchy Problem

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1 Introduction

We will first prove several results of the contraction mapping. With the tool of the Lipschitz condition and some analysis of the convergence of the Picard approximations, we will construct a complete metric space M and a contraction mapping based on the space. The Existence and Uniqueness Theorem will then follow from the fixed point of the contraction mapping.

2 Contraction Mapping

We will first define two following definitions.

Definition 2.1 Let $A : M \to M$ be a mapping of a metric space (M, ρ) into it self. The mapping M is called a *contraction* if there exists a constant λ with $0 < \lambda < 1$ such that

$$\rho(Ax, Ay) \leq \lambda\rho(x, y) \quad \forall x, y \in M \tag{1}$$

Directly from the definition, any contraction map satisfies the Lipschitz condition and hence is Lipschitz continuous (introduced in the section 3).

Definition 2.2 A point $x \in M$ is called a *fixed point* of a mapping $A : M \to M$ if $Ax = x$.

The contraction mapping theorem then states that a contraction mapping of a complete metric space has a unique fixed point:

Theorem 2.1 (The Contraction Mapping Theorem) Let $A : M \to M$ be a contraction mapping of a complete metric space M into itself. Then A has one and only one fixed point. For any point x in M, the sequence of images of the point x under applications of A

$$x,\ Ax,\ A^2x,\ A^3x,\ \dots \tag{2}$$

converges to the fixed point.

Proof. We will first prove the existence of a fixed point. Since ρ is a contraction, we can apply the inequality (1) for n times repeatedly,

$$\rho(A^n x, A^{n+1}x) \leq \lambda\rho(A^{n-1}x, A^n x) \leq \lambda^2\rho(A^{n-2}x, A^{n-1}x) \leq \cdots \leq \lambda^n\rho(x, Ax)$$

Hence, we have

$$\rho(A^n x, A^{n+1}x) \leq \lambda^n\rho(x, Ax)$$

From induction, we can further conclude that for $m \geq n \geq 1$

$$\rho(A^n x, A^m x) \leq \lambda^n\rho(x, A^{m-n}x)$$

We can then prove that the sequence in (2) is Cauchy using the result above and the triangle inequality.

$$\begin{aligned}
\rho(A^n x, A^m x) &\leq \lambda^n\rho(x, A^{m-n}x) \\
&\leq \lambda^n[\rho(x, Ax) + \rho(Ax, A^2x) + \cdots + \rho(A^{m-n-1}x, A^{m-n}x)] \\
&\leq \lambda^n\rho(x, Ax) \cdot \sum_{k=0}^{n-m-1} \lambda^k \\
&\leq \lambda^n \cdot \sum_{k=0}^{n-m-1} \lambda^k \cdot \rho(x, Ax) \\
&\leq \lambda^n \cdot \sum_{k=0}^{\infty} \lambda^k \cdot \rho(x, Ax)
\end{aligned}$$

As we know for $0 < \lambda < 1$, the geometric series converges, and we have

$$\rho(A^n x, A^m x) \leq \frac{\lambda^m}{1-\lambda}\rho(x, Ax)$$

Let $n, m \to \infty$, and it implies that $(A^n x)$ is Cauchy. Since M is complete, it converges to a limit, say

$$X := \lim_{n\to\infty} A^n x$$

As mentioned before, any contraction mapping is continuous, and we have

$$AX = A\lim_{n\to\infty} A^n x = \lim_{n\to\infty} A^{n+1}x = X$$

Therefore the limit X is a fixed point of A. To prove the uniqueness, suppose that X and Y are two fixed points, then

$$0 \leq \rho(X, Y) = \rho(AX, AY) \leq \lambda\rho(X, Y)$$

Since $\lambda < 1$, we have $\rho(X, Y) = 0$.

From the discussion above, we conclude the proof. □

3 The successive Approximations of Picard

In this section, we will first introduce the concept of Picard mapping. We will then construct a complete metric space such that the Picard mapping is a contraction, and we can then apply the contraction mapping theorem. The existence of the solution to our ODE is ensured by the existence of the fixed point of the contraction.

3.1 Picard mapping and approximations

Definition 3.1.1 Consider the differential equation $\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x})$ defined by the vector field $\boldsymbol{v}$ in some domain of the extended phase space $\mathbb{R}^{n+1}$. We define the *Picard mapping* to be the mapping A that takes the function $\boldsymbol{\varphi} : t \mapsto \boldsymbol{x}$ to the function $A\boldsymbol{\varphi} : t \mapsto \boldsymbol{x}$ where

$$(A\boldsymbol{\varphi})(t) = \boldsymbol{x_0} + \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{\varphi}(\tau))d\tau \tag{3}$$

It is easy to see that the equation (3) implies

$$(A\boldsymbol{\varphi})'(t) = \boldsymbol{v}(t, \boldsymbol{\varphi}(t)) \tag{4}$$

So geometrically, the tangent of the new curve $A\boldsymbol{\varphi}$ for each t is equal to the value of the given vector field at $(t, \boldsymbol{\varphi}(t))$, namely, on the graph of φ. Also, if these vectors lie on the graph of $A\boldsymbol{\varphi}$, we then have

$$(t, \boldsymbol{\varphi}(t)) = (t, (A\boldsymbol{\varphi})(t)) \quad \Longrightarrow \quad \boldsymbol{\varphi}(t) = (A\boldsymbol{\varphi})(t)$$

and from equation (4)

$$\boldsymbol{\varphi}'(t) = (A\boldsymbol{\varphi})'(t) = \boldsymbol{v}(t, \boldsymbol{\varphi}(t))$$

It means that $A\boldsymbol{\varphi}$ is a solution to the differential equation, where it coincides with the value of φ (fixed point!).

Motivated by the contraction mapping theorem, we define the sequence of *Picard approximations*

$$\boldsymbol{\varphi}, A\boldsymbol{\varphi}, A^2\boldsymbol{\varphi}, \ldots$$

starting with $\boldsymbol{\varphi} = \boldsymbol{x_0}$. Now, we will construct a complete metric space where the Picard mapping defined before is a contraction. In that case, we can prove the convergence of this Picard approximations sequence.

3.2 The Lipschitz Condition

Definition 3.2.1 Let $A : M_1 \to M_2$ be a mapping of two metric spaces (M_1, ρ_1) and (M_2, ρ_2). Let L be a positive real number. The mapping A *satisfies a Lipschitz condition with constant* L (written $A \in \text{Lip}L$) if the following inequality holds true for all $x, y \in M_1$:

$$\rho_2(Ax, Ay) \leq L\rho_1(x, y) \tag{5}$$

We can also phase the Lipschitz condition as the mapping increases the distance between any two points of M_1 by a factor of at most L.

Now, let $\boldsymbol{f} : U \to \mathbb{R}^n$ be a smooth mapping of class C^r with $r \geq 1$ from the domain U of an Euclidean space $\mathbb{R}^m$ to another Euclidean space $\mathbb{R}^n$. As the tangent space to the Euclidean space at each point has its own natural Euclidean structure, the derivative of $\boldsymbol{f}$ at a point $\boldsymbol{x} \in U \subset \mathbb{R}^m$

$$\boldsymbol{f}_{*\boldsymbol{x}} : T_{\boldsymbol{x}}\mathbb{R}^m \to T_{\boldsymbol{f}(\boldsymbol{x})}\mathbb{R}^n$$

is a linear operator from one Euclidean space into another. We then have the following theorem:

Theorem 3.2.1 A continuously differentiable mapping $\boldsymbol{f}$ satisfies a Lipschitz condition on each convex compact subset V of the domain U with a constant L equal to the supremum of the derivative of the function f on V:

$$L = \sup_{\boldsymbol{x} \in V} |\boldsymbol{f}_{*\boldsymbol{x}}|$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in V$ and connect them with a line segment by the convex combination:

$$\boldsymbol{z}(t) = (1-t)\boldsymbol{x} + t\boldsymbol{y}, \quad 0 \leq t \leq 1$$

Since $\boldsymbol{f}$ is continuously differentiable, the derivative of $\boldsymbol{f}$ is continuous, and its maximum value is attained on the compact set V. By the Barrow formula,

$$\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) = \int_0^1 \frac{d}{dt}(\boldsymbol{f}(\boldsymbol{z})(\tau))d\tau = \int_0^1 \boldsymbol{f}_{*\boldsymbol{z}(\tau)}\frac{d\boldsymbol{z}(\tau)}{dt}d\tau = \int_0^1 \boldsymbol{f}_{*\boldsymbol{z}(\tau)}(\boldsymbol{y} - \boldsymbol{x})d\tau$$

Therefore, we have

$$|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})| = \left|\int_0^1 \boldsymbol{f}_{*\boldsymbol{z}(\tau)}(\boldsymbol{y} - \boldsymbol{x})d\tau\right| \leq \int_0^1 |\boldsymbol{f}_{*\boldsymbol{z}(\tau)}(\boldsymbol{y} - \boldsymbol{x})|d\tau \leq \sup_{\boldsymbol{x} \in V} |\boldsymbol{f}_{*\boldsymbol{x}}||\boldsymbol{y} - \boldsymbol{x}|$$

□

3.3 The Metric Space M

Let's consider again the differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}) \tag{6}$$

Let $\boldsymbol{v}$ be differentiable of class C^r with $r \geq 1$ in a domain U of the extended phase space $U \subset \mathbb{R} \times \mathbb{R}^n$. Fix a Euclidean structure in $\mathbb{R}^n$, and hence also in $T_{\boldsymbol{x}}\mathbb{R}^n$. Fix a point $(t_0, \boldsymbol{x_0}) \in U$, and we will study the behavior within a small neighborhood of this point. For sufficient small a and b, define the cylinder

$$\mathcal{C} := \{t, \boldsymbol{x} : |t - t_0| \leq a,\ |\boldsymbol{x} - \boldsymbol{x_0}| \leq b\}$$

Since this cylinder is compact, the suprema of $\boldsymbol{v}$ is attained, we define

$$C := \sup_{t, \boldsymbol{x} \in \mathcal{C}} |\boldsymbol{v}(t, \boldsymbol{x})|$$

We now consider the cone K_0 with the center vertex $(t_0, \boldsymbol{x_0})$, aperture C, and height a':

$$K_0 := \{t, \boldsymbol{x} : |t - t_0| \leq a',\ |\boldsymbol{x} - \boldsymbol{x_0}| \leq C|t - t_0|\}$$

Note that the reason why we choose the aperture as C is to let solution lie inside the cone K_0. To do this, we first notice that the cones K_0 lies inside the cylinder $\mathcal{C}$ for sufficiently small a'. Explicitly, we simply let

$$a' \leq b/C \implies Ca' \leq b \quad \text{and} \quad |\boldsymbol{x} - \boldsymbol{x_0}| \leq C|t - t_0| \leq Ca' \leq b$$

Moreover, we can consider a parallel translation of K_0 as $K_{\boldsymbol{x}}$ by translating the vertex $(t_0, \boldsymbol{x_0})$ to $(t_0, \boldsymbol{x})$ with $|\boldsymbol{x} - \boldsymbol{x_0}| \leq b'$. In this case, the new cone $K_{\boldsymbol{x}}$ also lies inside the cylinder $\mathcal{C}$ for sufficiently small a' and b' similarly, and we will assume this for later on. The solution $\boldsymbol{\varphi}$ of Eq.(6) with initial condition $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}$ can be considered in the form

$$\boldsymbol{\varphi}(t) = \boldsymbol{x} + \boldsymbol{h}(t, \boldsymbol{x})$$

where $\boldsymbol{h}(t, \boldsymbol{x})$ is the varied vertical distance between the curve and the value $\boldsymbol{x}$. The solution lies inside the cone $K_{\boldsymbol{x}}$ subject to the initial conditions mentioned above.

Now we will construct the complete metric space. Define M as the set of all continuous mappings $\boldsymbol{h}$ of the cylinder $\mathcal{C}'$

$$\mathcal{C}' = \{t, \boldsymbol{x} : |t - t_0| \leq a',\ |\boldsymbol{x} - \boldsymbol{x_0}| \leq b'\}$$

with an additional condition

$$|\boldsymbol{h}(t, \boldsymbol{x})| \leq C|t - t_0| \tag{7}$$

and the initial condition is $\boldsymbol{h}(t, \boldsymbol{x} = 0)$. Define a metric ρ as

$$\rho(\boldsymbol{h}_1, \boldsymbol{h}_2) = \|\boldsymbol{h}_1 - \boldsymbol{h}_2\| = \max_{t, \boldsymbol{x} \in \mathcal{C}'} |\boldsymbol{h}_1(t, \boldsymbol{x}) - \boldsymbol{h}_2(t, \boldsymbol{x})|$$

The conditions of a metric are easy to verified. We will then prove the following theorem.

Theorem 3.3.1 The set M, endowed with the metric ρ, is a complete metric space.

Proof. For a sequence of functions $\{\boldsymbol{h}_n\}$ in M, the limit function is also in M as

$$|\boldsymbol{h}_n(t,\boldsymbol{x})| \leq C|t-t_0| \quad \Longrightarrow \quad \lim_{n\to\infty} |\boldsymbol{h}_n(t,\boldsymbol{x})| \leq C|t-t_0| \quad \Longrightarrow \quad |\boldsymbol{h}(t,\boldsymbol{x})| \leq C|t-t_0|$$

A uniformly convergent sequence of continuous functions converges to a continuous function. Then every Cauchy sequence in M converges to a limit in M, and we can conclude the proof. □

3.4 The contraction Mapping

Let (M,ρ) be the complete metric space we construct above. Now we define a mapping $A : M \to M$ as

$$(A\boldsymbol{h})(t,\boldsymbol{x}) = \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{x} + \boldsymbol{h}(\tau,\boldsymbol{x}))d\tau \tag{8}$$

From the inequality (7), the point $(\tau, \boldsymbol{x} + \boldsymbol{h}(\tau,\boldsymbol{x}))$ belongs to the cone $K_{\boldsymbol{x}}$ as

$$|(\boldsymbol{x} + \boldsymbol{h}(\tau,\boldsymbol{x})) - \boldsymbol{x}_0| \leq |\boldsymbol{x} - \boldsymbol{x}_0| + |\boldsymbol{h}(\tau,\boldsymbol{x})| \leq b' + C|t-t_0|$$

and consequently it belongs to the domain of the vector field $\boldsymbol{v}$. It is also worth noting that this is another form of the Picard mapping in equation (3), but we are now instead looking for a solution in the form of $\boldsymbol{x} + \boldsymbol{h}$. We claim that the mapping A is a contraction mapping with the following condition:

Theorem 3.4.1 If the quantity a' is sufficiently small, then the formula (8) defines a contraction mapping of the space M into itself.

Proof. To show that A actually maps M to itself, one suffices to show that $AM \subseteq M$. Since the integral of a continuous function that depends continuously on a parameter is continuously dependent on the parameter and on the upper limit of integration, then the function $A\boldsymbol{h}$ is continuous. To prove that the function $A\boldsymbol{h}$ is in M, verify the inequality (7) from the definition of C

$$|(A\boldsymbol{h})(t,\boldsymbol{x})| = \left| \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{x} + \boldsymbol{h}(\tau,\boldsymbol{x}))d\tau \right| \leq \left| \int_{t_0}^{t} C d\tau \right| \leq C|t-t_0|$$

To show that it is indeed a contraction, we recall that $\boldsymbol{h}$ is simply the varied vertical distance between the curve and the value $\boldsymbol{x}$. We can then estimate the value of $A\boldsymbol{h}_1 - A\boldsymbol{h}_2$ as

$$(A\boldsymbol{h}_1 - A\boldsymbol{h}_2)(t,\boldsymbol{x}) = \int_{t_0}^{t} \boldsymbol{v}_1 - \boldsymbol{v}_2 d\tau$$

where

$$\boldsymbol{v}_i = \boldsymbol{v}(\tau, \boldsymbol{x} + \boldsymbol{h}_i(\tau,\boldsymbol{x})), i = 1, 2$$

For a fixed τ, the function $\boldsymbol{v}(\tau, \boldsymbol{x})$ is continuously differentiable and satisfies a Lipschitz condition, say L, on a convex compact set(cylinder $\mathcal{C}'$). From the Theorem 3.2.1, we have

$$\begin{aligned}
|(A\boldsymbol{h}_1 - A\boldsymbol{h}_2)(t, \boldsymbol{x})| &\leq \left| \int_{t_0}^{t} \boldsymbol{v}_1(\tau) - \boldsymbol{v}_2(\tau) d\tau \right| \\
&\leq \left| \int_{t_0}^{t} L|\boldsymbol{h}_1(\tau, \boldsymbol{x}) - \boldsymbol{h}_2(\tau, \boldsymbol{x})| d\tau \right| \\
&\leq La'|\boldsymbol{h}_1 - \boldsymbol{h}_2| \leq La' \, \|\boldsymbol{h}_1 - \boldsymbol{h}_2\|
\end{aligned}$$

With a' sufficiently small such that $La' < 1$, the mapping is a contraction. □

4 The Existence and Uniqueness Theorem

We now obtain all the tools that we will need to prove the theorem. Here we will point out how the Picard mapping and the contraction mapping A are connected. The rigorous statement of the theorem is the following.

Theorem 4.1 (The Existence and Uniqueness Theorem) Consider the differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(t, \boldsymbol{x}) \tag{9}$$

Let $\boldsymbol{v}$ be continuously differentiable in the neighborhood of the point $(t_0, \boldsymbol{x}_0)$ in a domain U of the extended phase space $U \subset \mathbb{R} \times \mathbb{R}^n$. Then there is a neighborhood of the point t_0 such that a unique solution of Eq.(9) is defined in this neighborhood with the initial condition $\boldsymbol{\varphi}(t_0) = \boldsymbol{x}$, where $\boldsymbol{x}$ is any point sufficiently close to $\boldsymbol{x}_0$. Moreover, this solution depends continuously on the initial point $\boldsymbol{x}$.

Proof. Let M be the metric space we defined in section 3.3 and A be the contraction mapping in section 3.4. From the Contraction Mapping theorem, we know that the mapping A has a fixed point, say $\boldsymbol{h} \in M$. It then satisfies

$$\boldsymbol{h}(t, \boldsymbol{x}) = (A\boldsymbol{h})(t, \boldsymbol{x}) = \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{x} + \boldsymbol{h}(\tau, \boldsymbol{x})) d\tau$$

We define

$$\boldsymbol{\varphi}(t, \boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{h}(t, \boldsymbol{x}) = \boldsymbol{x} + \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{x} + \boldsymbol{h}(\tau, \boldsymbol{x})) d\tau = \boldsymbol{x} + \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{\varphi}(\tau, \boldsymbol{x})) d\tau$$

Since $\boldsymbol{h} \in M$, the function $\boldsymbol{g}$ is continuous. To verify that it is a solution, for fixed $\boldsymbol{x}$, simply compute

$$\frac{\partial \boldsymbol{\varphi}(t, \boldsymbol{x})}{\partial t} = \boldsymbol{v}(\tau, \boldsymbol{\varphi}(\tau, \boldsymbol{x})) \quad \text{by Barrow formula}$$

with the initial condition

$$\boldsymbol{\varphi}(t_0, \boldsymbol{x}) = \boldsymbol{x} + \int_{t_0}^{t_0} \boldsymbol{v}(\tau, \boldsymbol{\varphi}(\tau, \boldsymbol{x})) d\tau = \boldsymbol{x}$$

Therefore it is a solution to the ODE with the initial condition $\boldsymbol{\varphi}(t_0, \boldsymbol{x}_0 = \boldsymbol{x})$.

Them proof of the uniqueness is very similar to the last part of the Theorem 3.4.1 It can also be shown by setting $b' = 0$ in M, and the uniqueness of the solution follows from the uniqueness of the fixed point in the contraction mapping theorem. Here we will represent the former proof as follows. Let two solutions be

$$\boldsymbol{\varphi}_i(t) = \boldsymbol{x} + \int_{t_0}^{t} \boldsymbol{v}(\tau, \boldsymbol{\varphi}_i(\tau))d\tau, i = 1, 2$$

Suppose that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are two solutions with the same initial condition $\boldsymbol{\varphi}_1(t_0) = \boldsymbol{\varphi}_2(t_0)$ with in a neighborhood of the point t_0 for $|t - t_0| \leq \alpha$. Let $0 < \alpha' < \alpha$ and define $\|\boldsymbol{\varphi}\| = \max_{|t-t_0|\leq\alpha'} |\boldsymbol{\varphi}(t)|$. For a fixed τ, the function $\boldsymbol{v}(\tau, \boldsymbol{x})$ is continuously differentiable and satisfies a Lipschitz condition, say L, on a convex compact set(cylinder $\mathcal{C}'$). From the Theorem 3.2.1, we have

$$\begin{aligned}
\|\boldsymbol{\varphi}_1(t) - \boldsymbol{\varphi}_2(t)\| &\leq \left\| \int_{t_0}^{t} \boldsymbol{v}_1(\tau, \boldsymbol{\varphi}_1(\tau)) - \boldsymbol{v}_2(\tau, \boldsymbol{\varphi}_2(\tau))d\tau \right\| \\
&\leq \left\| \int_{t_0}^{t} L|\boldsymbol{\varphi}_1(\tau) - \boldsymbol{\varphi}_2(\tau)|d\tau \right\| \\
&\leq La' \|\boldsymbol{\varphi}_1(t) - \boldsymbol{\varphi}_2(t)\|
\end{aligned}$$

With a' sufficiently small such that the two solutions lie inside the cylinder and $La' < 1$, we obtain that $\|\boldsymbol{\varphi}_1(t) - \boldsymbol{\varphi}_2(t)\| = 0$. It implies that the solutions coincide in some neighborhood of the point t_0, and we can finish the proof. $\square$