Lecture note 2

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Lecture 6

1 Approximate L^p functions

It is possible to use simple functions to approximate L^p functions, and we have the following theorem

Theorem 2.1.1 Simple functions are dense in $L^p(X)$ for $1 \leq p \leq \infty$, where the simple functions are defined of the form

$$\phi(x) = \sum_{i=1}^{N} a_i \chi_{A_i}$$

with $A_i \subseteq X$ measurable finitely, $\mu(A_i) < \infty$, and $|a_i| < \infty$.

Proof. Let $f \in L^p(X)$, WLOG, assume that $f \geq 0$, (or we can let $f = f^+ - f^-$). We will construct a sequence of simple functions $S_1 \leq S_2 \leq S_3 \leq \cdots \leq f$. First define

$$E_{k,n} := \{x \in X : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}$$

where $1 \leq k \leq (n+1)2^n$. Notice that if n goes to infinity, these sets cover the entire space since the length of the interval $1/2^n$ goes to zero. Namely,

$$\bigcup_{n \geq 1} \bigcup_{k \geq 1}^{(n+1)2^n} E_{k,n} = X$$

Now we can define the simple functions as

$$S_n = \sum_{k=1}^{(n+1)2^n} \frac{k-1}{2^n} \chi_{E_{k,n}}$$

with the fact that $S_n \leq S_{n+1}$. The indicator function indicates the interval location of the function values, and coefficients are used to approximate the function values. Therefore, $S_n \to f$ *a.e.*, and $f \in L^p$ implies $S_n \in L^p$. If $1 \leq p < \infty$, we have

- $|S_n - f|^p \leq 2^p f^p \in L^1$
- $|S_n - f|^p \to 0$ *a.e.*

By dominated convergence theorem, we have

$$\lim_{n\to\infty} \|S_n - f\|_{L^p} = \left\| \lim_{n\to\infty} (S_n - f) \right\|_{L^p} = 0$$

And for $p = \infty$, we obtain the uniform convergence as

$$\|S_n - f\|_{L^\infty} \leq \frac{1}{2^n} \text{ independent of } x \quad \Longrightarrow \quad S_n \to f \text{ in } L^p$$

as desired. □

We then will explore to see the approximation in the $\mathbb{R}^n$ spaces.

Definition 2.1.2 Let $(\mathbb{R}^n, \sigma, \lambda)$ be the Lebesgue measure space, we define the *step function* as the form

$$\varphi(x) = \sum_{i=1}^{N} a_i \chi_{Q_i}$$

where Q_i are pairwise disjoint cubes of the form

$$Q_i = \prod_{i=1}^{n} [\alpha_i, \alpha_i + l] \quad \alpha_i, l \in \mathbb{R}^+$$

Here are several properties about the step functions

Proposition 2.1.3 Properties of the step functions

- {Step functions} $\subseteq$ {Simple functions}
- Step functions are dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$
- It is *not* dense in $L^\infty(\mathbb{R}^n)$.

Proof. The first property is easy to verify as one can write any step function as a simple function. The converse is obviously not true and the examples could be the characteristic function of the Cantor set or the characteristic function of the set of all rational numbers.

We will prove the third property first. We can take the space as $L^\infty(\mathbb{R})$. There exists a set $A \subseteq \mathbb{R}$ Lebesgue measurable s.t. both sets $A \cap I$ and $A^c \cap I$ have positive measure for any non-empty open intervals I.[1] We claim the following:

Claim: χ_A can not be approximated uniformly by Step functions

Now it suffices to prove the claim, and we can define some sequences of step functions

$$\varphi(x) = \sum_{i=1}^{\infty} c_n \chi_{I_i}$$

Then

$$\|\varphi - \chi_A\|_{L^\infty} \geq \|c_i \chi_{I_i} - \chi_A\|_{L^\infty} \geq \max\{|1 - c_i|, |c_i|\} \geq \frac{1}{2}$$

To prove the second property, it suffices to prove that the indicator functions can be approximated by step functions, namely

$$\chi_A \sim \sum_{i=1}^{N} a_i \chi_{Q_i}$$

If fact, we can set the coefficients to be 1 as

$$\chi_A \sim \sum_{i=1}^{N} \chi_{Q_i}$$

and we can conclude the proof from the Theorem 2.1.1.

We will state the following two lemmas without proving.

Lemma 2.1.4 Suppose that A is a measurable subset of $\mathbb{R}^n$. Then for every $\epsilon > 0$, there exists an open set $\mathcal{O} \subseteq \mathbb{R}^n$ with $A \subset \mathcal{O}$ and $\lambda(\mathcal{O} - A) \leq \epsilon$.[2]

Lemma 2.1.5 Every open subset $\mathcal{O} \subseteq \mathbb{R}^n$, $n \geq 1$, can be written as a countable union of almost disjoint closed cubes, where we define that a union of cubes is said to be *almost disjoint* if the interiors of the cubes are disjoint.[3]

The lemma 2.1.4 implies that we can use an open set to approximate a measurable set as

$$\|\chi_{\mathcal{O}} - \chi_A\|_{L^p} < \epsilon$$

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[1] One example of this kind of set can be constructed within the interval $[0, 1]$ as following: *"Consider a Cantor-like set of positive measure, and add in each of the intervals that are omitted in the first of its construction, and other Cantor-like set. Continue this procedure."* The characteristics function of this set actually has the property that whenever $g(x) = f(x)$ *a.e.*, then g must be discontinuous at every point in $[0, 1]$. *Reference: Exercise 1.36, P45, Real Analysis by Stein and Shakarchi.*

[2] This is called *the Borel regularity of Lebesgue measure.* The proof of this fact can be referred as the Theorem 3.4 at the page P21 in the book *Real Analysis by Stein and Shakarchi.*

[3] The proof of this fact can be referred as the Theorem 1.4 at the page P7 in the book *Real Analysis by Stein and Shakarchi.*

and the question becomes to

$$\chi_{\mathcal{O}} \sim \sum_{i=1}^{N} \chi_{Q_i}$$

Therefore, the proof is finished from the Lemma 2.1.5. $\square$

We have shown that the simple functions and the step functions are both dense in $L^p(\mathbb{R}^n)$. Here we come to the fact that the continuous functions are dense in $L^p(\mathbb{R}^n)$. More important, the continuous functions with compact support are dense in $L^p(\mathbb{R}^n)$.

Theorem 2.1.6 If $X \subseteq \mathbb{R}^n$, $1 \leq p < \infty$, then

$$C_c(X) = \{\text{all continuous functions on } X \text{ with compact support}\}$$

is dense in $L^p(X, \lambda)$.

Scratch proof. We want to prove that: $\forall f \in L^p$, $\forall \epsilon > 0$, there exists $g \in C_c$ s.t. $\|f - g\|_p < \epsilon$. We will let f be a characteristics function, a simple function, and then a L^p function.

Proof. Consider the following:

1. When $f = \chi_A$ for some bounded measurable set $A \subseteq X \subseteq \mathbb{R}^n$.

   Given $\epsilon > 0$, by the Borel regularity of Lebesgue measure[4], there exists a bounded open set G and a compact set K s.t.

   $$K \subset A \subset G \quad \text{and} \quad \lambda(G \setminus K) < \epsilon$$

   Let $g \in C_c(\mathbb{R}^n)$ be a Urysohn function s.t. $g \in [0, 1]$ and

   $$g = \begin{cases} 1, & x \in K \\ 0, & x \in G^c \end{cases}$$

   One example of g could be the following:

   $$g(x) = \frac{d(x, G^c)}{d(x, k) + d(x, G^c)}$$

   where the distance function is

   $$d(x, F) = \inf\{|x - y : y \in F\}$$

   Therefore,

   $$\left\| \chi_A - g\Big|_X \right\|_p = \left(\int_{(G \setminus K) \cap X} |\chi_A - g|^p \, d\lambda \right)^{1/p} \leq (\lambda(G \setminus K))^{1/p} < \epsilon^{1/p}$$

[4] See the footnote 2 for more.

2. When $f = \sum_{i=1}^{N} a_i \chi_{A_i}$, $a_i > 0$, is a non-negative simple function with $A_i \subseteq X$ bounded.

   For each i, there exists $g_i \in C_c(X)$ s.t.

$$\|\chi_{A_i} - g_i\|_p \leq \frac{\epsilon}{N|a_i|}$$

   It implies

$$\left\| f - \sum_{i=1}^{N} a_i g_i \right\|_p \leq \sum_{i=1}^{N} |a_i| \cdot \|\chi_{A_i} - g_i\|_p < \epsilon$$

3. When $f \in L^p$.

   Since the simple function is dense in L^p and also continuous, we can conclude the proof.

□

Lecture 7

2 Dual Characterization of $L^p(X)$

We have learned to compute the L^p norm of a function in two ways before. One is the standard definition, and the other one is to use the distribution function. Here we will represent the third way to compute the L^p norm.

Theorem 2.2.1 Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. When $1 \leq p < \infty$.

   We can compute

$$\|f\|_{L^p(X)} = \sup_{\|g\|_{L^q}=1} \int_X f \cdot g \, d\mu$$

   where $1/p + 1/q = 1$.

2. When $p = \infty$.

   If X is σ-finite, meaning that

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad \mu(X_i) < \infty$$

   then

$$\|f\|_{L^\infty(X)} = \sup_{\|g\|_{L^1}=1} \int_X f \cdot g \, d\mu$$

Proof. From the Holder's inequality, it is clear that

$$\sup_{\|g\|_{L^q}=1} \int_X f \cdot g \, d\mu \leq \sup_{\|g\|_{L^q}=1} \|f\|_{L^p(X)}$$

now it suffices to check the other direction. To do this, we will pick $g \in L^q$ with $\|g\|_{L^q} = 1$ and that

$$\int f \cdot g \, d\mu = \|f\|_{L^p}$$

We now consider the following two situations,

1. When $q = \infty$.

   Pick $g \in L^q$ with $\|g\|_{L^q} = 1$ and that

   $$\int f \cdot g \, d\mu = \|f\|_{L^1} = \int |f| \quad \Longrightarrow \quad g = \text{sign } f$$

2. When $1 \leq q < \infty$. Pick $g \in L^q$ with $\|g\|_{L^q} = 1$ and that

   $$\int f \cdot g \, d\mu = \|f\|_{L^p} = \left(\int |f|^p \right)^{1/p}$$

   Note that $q = p/(p-1)$, we can define g such that

   $$g = \frac{\text{sign } |f|^{p-1}}{\||f|^{p-1}\|_{L^q}} \in L^q$$

   and one can check that

   $$\int f \cdot g \, d\mu = \int \frac{|f|^p}{\||f|^{p-1}\|_{L^q}} = \int |f|^p \Big/ \left(\int |f|^p \right)^{1/q} = \left(\int |f|^p \right)^{1/p} = \|f\|_{L^p}$$

   as desired.

To prove the case when $p = \infty$, the restriction of the space X to be σ-finite is needed because

It can also be proved from both direction, and again from the Holder's inequality

$$\sup_{\|g\|_{L^1}=1} \int_X f \cdot g \, d\mu \leq \sup_{\|g\|_{L^1}=1} \|f\|_{L^\infty(X)}$$

For the other direction, we first assume that $\|f\|_{L^\infty(X)} \neq 0$, or the proof is trivial. Fix $\epsilon > 0$ and define the set

$$A_\epsilon \coloneqq \{x \in X : |f(x)| \geq \|f\|_{L^\infty} - \epsilon\}$$

Since X is σ-finite, it means that

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad \mu(X_i) < \infty$$

We define

$$g_\epsilon := \frac{\text{sign } f \cdot \chi_{A_\epsilon \cap X_n}}{\mu(A_\epsilon \cap X_n)} \quad \text{and} \quad g \in L^1 \text{ s.t. } \|g\|_{L^1} = 1$$

for all n where $\mu(A_\epsilon \cap X_n) \neq 0$. Now we compute

$$\int_X f \cdot g = \int_{A_\epsilon \cap X_n} \frac{\text{sign } f \cdot f}{\mu(A_\epsilon \cap X_n)} = \int_{A_\epsilon \cap X_n} \frac{|f|}{\mu(A_\epsilon \cap X_n)} \geq \frac{\mu(A_\epsilon \cap X_n)}{\mu(A_\epsilon \cap X_n)}(\|f\|_{L^\infty} - \epsilon) \geq \|f\|_{L^\infty} - \epsilon$$

for all $\epsilon > 0$. Then the proof is complete. □

We will give some definitions before we get into the Riesz Representation Theorem.

Definition 2.2.2 A *bounded linear functional* l is a map $l : L^p(X) \to \mathbb{R}$ with the properties

$$l(af + bg) = al(f) + b(g) \quad \text{and} \quad |l(f)| \leq C \|f\|_{L^p}$$

for some constants A, B, and C independent of f. The *operator norm* of such a functional is defined as

$$\|l\| := \inf\{C : |l(f)| \leq C \|f\|_{L^p} \quad \forall f \in L^p\}$$

We will now investigate the space of the bounded linear functionals.

Definition 2.2.3 The set of all the bounded linear functions on L^p is called the *dual* of $L^p(X)$, denoted as $(L^p(X))^*$.

Before introducing the Riesz-Representation Theorem, we will first represent the following Theorem:

Theorem 2.2.4 (Radon-Nikodym Theorem) Suppose that $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ are finite measure spaces. Suppose that ν is absolutely continuous respective to μ, $\nu \ll \mu$, namely,

$$\mu(A) = 0 \quad \Longrightarrow \quad \nu(A) = 0$$

Then there exists a measurable, non-negative, integrable function f such that

$$\nu(A) = \int_A f \, d\mu \quad \forall A \in \mathcal{A}$$

Proof. We need the Riesz-Representation Theorem for Hilbert Space, stated below:

Theorem 2.2.5 (Riesz-Representation Theorem for Hilbert Space) Let $\mathcal{H}$ be a Hilbert space and let $l \in \mathcal{H}^*$. Then there exists a unique $g \in \mathcal{H}$ such that $l(x) = \langle x, g \rangle$.

Define a new measure $\pi := \mu + \nu$ and consider the Hilbert space $L^2(X, \mathcal{A}, \pi)$. We denote

$$l(f) = \int_X f \, d\nu$$

and we will prove that $l \in (L^2(X, \mathcal{A}, \pi))^*$. By the Holder's inequality

$$|l(f)| = \left| \int_X f \; d\nu \right| \leq \int_X |f| \; d\nu \leq \int_X |f| d\pi \leq \|f\|_{L^2} (\pi(x))^{1/2} < (\pi(x))^{1/2} < \infty$$

from the fact that μ and ν are both finite measure spaces.

From the Theorem 2.2.6, there exists a unique $g \in (L^2(X, \mathcal{A}, \pi))^*$ such that for all f

$$\int_X f \; d\nu = \int_X f \cdot g \; d\pi = \int_X f \cdot g \; (d\mu + d\nu)$$

It implies that

$$\int_X f(1-g) \; d\nu = \int_X f \cdot g \; d\mu$$

Define sets $B = \{x \in X : g(x) > 1\}$, $N = \{x \in X : g(x) < 0\}$, and $G = \{x \in X : 0 \leq g(x) < 1\}$ to analyze the behavior of g. Consider

1. To study the behavior of the set B, let $f = \chi_B$.

   We then have

$$\int_X \chi_B(1-g) \; d\nu = \int_X \chi_B \cdot g \; d\mu$$
$$\int_B \underbrace{1-g}_{\leq 0} \; d\nu = \int_B \underbrace{g}_{\geq 1} \; d\mu$$

   which implies that

$$\mu(B) = 0 \quad \Longrightarrow \quad \nu(B) = 0$$

2. To study the behavior of the set N, let $f = \chi_N$.

   We then have

$$\int_N \underbrace{1-g}_{\geq 1} \; d\nu = \int_N \underbrace{g}_{<0} \; d\mu$$

   which implies that

$$\mu(N) = 0 \quad \Longrightarrow \quad \nu(N) = 0$$

3. To study the behavior of the set G, let $f = \chi_G$.

   We then have

$$\int_G \underbrace{1-g}_{[0,1)} \; d\nu = \int_G \underbrace{g}_{[0,1)} \; d\mu$$

   which is well-behaved.

Therefore, as we can write $X = B \cup N \cup G$, we have

$$\int_{X=B\cup N\cup G} f(1-g)d\nu = \int_G f(1-g)\ d\nu = \int_G f \cdot g\ d\mu$$

and for $A \in \mathcal{A}$, choose the arbitrary $f = \chi_A/(1-g)$ in the Riesz-Representation to obtain the required function $f' = \chi_A \cdot g/(1-g)$ in the Radon-Nikodym Theorem. Namely

$$\begin{aligned}\nu(A) = \int_X \chi_A\ d\nu &= \int_X \frac{\chi_A}{1-g}(1-g)\ d\nu \\ &= \int_G \frac{\chi_A}{1-g}(1-g)\ d\nu \\ &= \int_G \frac{\chi_A}{1-g} g\ d\mu \\ &= \int_A \frac{\chi_A \cdot g}{1-g}\ d\mu\end{aligned}$$

as desired. □

Lecture 8

We can now prove the Riesz-Representation Theorem for L^p.

Theorem 2.2.6 (Riesz-Representation Theorem for L^p) If $1 \leq p < \infty$ and X is σ-finite, then

$$(L^p(X))^* = L^q(X)$$

where $1/p + 1/q = 1$. Namely, let $l \in (L^p(X))^*$. There exists a unique $g \in L^q(X)$ such that

$$l = l_g := \int_X f \cdot g\ d\mu$$

Proof. Assume that $\mu(X) < \infty$ and that $p > 1$. Let $E \in \mathcal{A}$ and $\chi_E \in L^p(X)$. Define that

$$l(\chi_E) := \nu(E)$$

where ν is a bounded signal measure on $(X, \mathcal{A})$, which satisfies the following criteria

- ν takes on only one infinity ∞ or $-\infty$.
- $\nu(\varnothing) = 0$.
- ν is countably additive.

and $\nu \ll u$ from the fact that

$$\mu(E) = 0 \quad \implies \quad l(\chi_E) = l(0) = 0 = \nu(E)$$

By Radon-Nikodym Theorem, there exists a unique $g \in L^1(\mu)$ such that $\forall E \in \mathcal{A}$ and

$$\nu(E) = l(\chi_E) = \int_X \chi_E g \; d\mu$$

Define an arbitrary simple function

$$\phi := \sum_{n=1}^{N} a_n \chi_{E_n}$$

then

$$l(\phi) = \sum_{n=1}^{N} a_n l(\chi_E) = \sum_{n=1}^{N} a_n \int_X \chi_{E_n} g \; d\mu = \int_X \phi g \; d\mu$$

Notice that for any simple function ϕ, $\|\phi\|_{L^p} = 1$ and

$$\|g\|_{L^q(X)} = \sup_{\|\phi\|_{L^p}=1} \int_X \phi g \; d\mu = l(\phi) \leq \|l\| \, \|\phi\|_{L^p} \leq \|l\| < \infty.$$

implying that $g \in L^q(X)$. To push the approximation, for any simple function $\phi \in L^p(\mu)$, take a sequence of increasing simple functions $\phi_n \nearrow f$ with f positive (or write $f = f^+ - f^-$), and $\phi_{n+1} \geq \phi_n$ and $\phi_n \to f$ in L^p. Consider

$$|l(f) - l(\phi_n)| \leq \|l\| \, \|f - \phi_n\|_{L^p} \quad \implies \quad \lim_{n\to\infty} |l(f) - l(\phi_n)| = 0$$

Therefore

$$l(f) = \lim_{n\to\infty} l(\phi_n) = \lim_{n\to\infty} \int_X \phi_n g \; d\mu$$

Point out the following facts

- By Holder's inequality, $f \in L^p$ and $g \in L^q$

$$|\phi_n g| \leq fg \in L^1$$

- Since $\phi_n \to f$ in L^p

$$\phi_n g \to fg \quad a.e.$$

and we can apply the Dominated Convergence Theorem,

$$l(f) = \lim_{n\to\infty} \int_X \phi_n g \; d\mu = \int_X \lim_{n\to\infty} \phi_n g \; d\mu = \int_X fg \; d\mu$$

Since X is σ-finite, let

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad X_{n+1} \supseteq X_n$$

with $\mu(X_n) < \infty$. We have that

$$f_n = f\chi_{X_n} \quad \text{and} \quad f_n \to f \quad a.e.$$

Then for any n, $f_n \in L^p(X_n, \mu)$, from the before result, there exists a unique $g_n \in L^q(X_n, \mu)$ with $\text{supp}(g_n) \subseteq X_n$ such that

$$l(f_n) = \int_{X_n} f_n g_n \; d\mu = \int_X f_n g_n \; d\mu$$

It implies that

$$|l(f_n)| \leq \|l\| \, \|f\|_{L^p} \quad \implies \quad \|g_n\|_{L^q(X)} \leq \|l\|$$

We then define

$$g(x) := \lim_{n\to\infty} g_n(x)$$

Now, consider

$$|f_n \cdot g_n| \leq |f| \cdot |g| \in L^1 \quad \text{and} \quad |f_n \cdot g_n \to fg \quad a.e.$$

and by D.C.T

$$l(f) = \lim_{n\to\infty} l(f_n) = \lim_{n\to\infty} \int_X f_n g_n \; d\mu = \in_X f \cdot g \; d\mu$$

as desired. □

The Riesz-Representation Theorem for L^p simply tells us that there exists an isometry between the space $(L^p(X))^*$ and the space $L^q(X)$ for $1 \leq p < \infty$

$$(L^p(X))^* \cong L^q(X)$$

We also have the following implication that for $p = \infty$

$$(L^\infty(X))^\infty \supseteq L^1(X)$$

Notice there exist some bounded linear functionals that do not appear to be in the L^1 space. Here is an example:

Example 2.2.7 Let the space be $(L^p(\mathbb{R}))^*$ and the space of continuous functions of compact support $C_c(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$. Consider the bounded linear functional $l(f) = f(0)$ and a sequence of non-negative functions

$$f_n(x) = \begin{cases} -2^n x + 1 & \text{if } x \in [0, (1/2)^n] \\ 0 & \text{otherwise} \end{cases}$$

and it is a function consisting of a line segment from the point $(0, 1)$ to the point $((1/2)^n, 0)$ and zero anywhere else. The limit of these functions is

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

We then have

$$\lim_{n\to\infty}\int f_n g\ dx = 0$$

for all $g \in L^1$, but

$$l(f_n) = f_n(0) = 1 \neq 0$$

We will now introduce another type of convergence which is more easy to obtain and related to the bounded linear functionals.

Definition 2.2.8 (Weak Convergence) A sequence (x_n) in a Banach space B *converges weakly* to $x \in B$ if for any $f \in B^*$, we have

$$f(x_n) \to f(x)$$

It is always written as

$$x_n \rightharpoonup x$$

Example 2.2.9 The sequence $(\sin(n\pi x))$ converges weakly to zero in $L^2([0,1])$ because

$$\int_0^1 f(x)\sin(n\pi x)\ dx \to 0$$

as $n \to \infty$ for all $f \in L^2([0,1])$. The sequence cannot converge strongly to zero since $\|\sin(n\pi x)\| - 1/\sqrt{2}$ is bounded away from zero.

The strong convergence implies weak convergence, but the converse is not true on infinite-dimensional spaces.

Theorem 2.2.10 Suppose X is σ-finite, $1 \leq p < \infty$. If $f_n \to f$ in $L^p(X)$, then $f_n \rightharpoonup f$ in $L^p(X)$.

Proof. Consider

$$\int |f_n - f||g| \leq \|f_n - f\|_{L^p}\|g\|_{L^q} \to 0$$

then

$$\int f_n g \to \int fg$$

for any $g \in L^q$ as desired. □