Lecture note 1

Vincent Chen

March 2020

Without specifying, the following conditions apply to all the sections below: Given a measure space $(X, \mathcal{A}, \mu)$. With $D \in \mathcal{A}$, define a sequence of $\mathcal{A}$-measurable functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$.

Lecture 1:

1 Convergence Almost Everywhere (a.e.)

Definition 1.1.1 The following conditions are equivalent:

1. $f_n \to f$ almost everywhere.
2. There exists $A \subseteq D$ and $A \in \mathcal{A}$ measurable with $\mu(A) = 0$ measure zero such that $\forall x \in D \setminus A$, we have
$$f_n(x) \to f(x)$$
3. The set where $f_n(x)$ does not converge to $f(x)$ has measure zero.
$$\mu(\{x \in D : f_n(x) \not\to f(x)\}) = 0$$
4. The set where $f_n(x)$ does not converge to $f(x)$ for infinitely many n has measure zero. Namely, given $\epsilon > 0$
$$\mu(\{x \in D : |f_n(x) - f(x)| \geq \epsilon \quad \text{for infinitely many } n\}) = 0$$
5. Given $\epsilon > 0$, we define
$$E_n^\epsilon = \{x \in D : |f_n(x) - f(x)| \geq \epsilon\}$$

and write

$$E^\epsilon = \bigcap_{k=1}^{\infty} \bigcup_{n\geq k} E_n^\epsilon =: \limsup_{n\to\infty} E_n^\epsilon$$

where E^ϵ can be naturally defined as

$$E^\epsilon = \{x \in D : |f_n(x) - f(x)| \geq \epsilon \quad \text{for infinitely many } n\}$$

Now we need to have

$$\mu\left(E^\epsilon\right) = 0 \quad \forall \epsilon > 0$$

Several notes may be noticed:

- Any equivalent definition of a.e. above is simply a naturally rewritten form of the previous definition.
- The Borel-Cantelli lemma is highly related to the fifth definition, as mentioned in the section 6 later. It may be one criteria to verify the definition (5).
- Compare the following two definitions

  $$\limsup_{n\to\infty} E_n^\epsilon := \bigcap_{k=1}^{\infty} \bigcup_{n\geq k} E_n^\epsilon = \{x \in E_x \text{ for infinitely many } n\}$$

  and

  $$\liminf_{n\to\infty} E_n^\epsilon := \bigcup_{k=1}^{\infty} \bigcap_{n\geq k} E_n^\epsilon = \{x \in E_x \text{ for all but finitely many } n\}$$

  We directly have

  $$\liminf_{n\to\infty} E_n^\epsilon \subseteq \limsup_{n\to\infty} E_n^\epsilon$$

2 Convergence Almost Uniformly

Definition 1.2.1 We say that

$$f_n \to f \text{ almost uniformly}$$

if and only if for any $\epsilon > 0$, there exists $A_\epsilon \subseteq D$ and $A_\epsilon \in \mathcal{A}$ measurable with $\mu(A_\varepsilon) < \epsilon$ such that

$$f_n \to f \text{ uniformly on } D \setminus A_\epsilon = A_\epsilon^c$$

Note that one can write out the definition of uniformly convergence as: Given $\delta > 0$, there exists $N \in \mathbb{N}_{\geq 1}$ such that

$$|f_n - f| < \delta$$

for all $n > \mathbb{N}_{\geq N}$ on $D \setminus A_\epsilon$.

3 Convergence in Measure (Probability)

Definition 1.3.1 We say that

$$f_n \xrightarrow{\mu} f \text{ in measure}$$

if and only if for any $\epsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in D : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

4 L^p Convergence

Definition 1.4.1 We say that

$$f_n \to f \text{ in } L^p$$

for $1 \leq p \leq \infty$ if and only if

$$\lim_{n\to\infty} \int_D |f_n - f|^p d\mu = 0$$

5 Relations between convergences

We have the following relationships

1. Convergence almost uniformly implies convergence almost everywhere.

   Scratch Proof. We will take a sequence of sets $A_{1/n}$ and use the infinite intersection of them to reach the set of measure zero.

   Proof. Suppose that $f_n \to f$ almost uniformly. From the definition given above, let $\epsilon = 1/n$ for $n \in \mathbb{Z}_{\geq 1}$. The definition told us that for any $\epsilon = 1/n > 0$, there exists a set $A_{1/n}$ with $\mu(A_{1/n}) < 1/n$ such that

   $$f_n \to f(x) \text{ uniformly on } A^c_{1/n}$$

   Since the countable intersection of measurable sets is measurable, define

   $$A = \bigcap_{n=1}^{\infty} A_{1/n}$$

   we then have

   $$\mu(A) \leq \mu(A_{1/n}) < 1/n \quad \Longrightarrow \quad \mu(A) = 0 \text{ as } n \to \infty$$

By taking the complement of A, we have

$$A^c = \left(\bigcap_{n=1}^{\infty} A_{1/n} \right)^c = \bigcup_{n=1}^{\infty} A_{1/n}^c$$

As for every $x \in A^c$, it belongs to $A_{1/N}$ for some $N \in \mathbb{Z}_{\geq 1}$, and we have the uniform convergence hence the pointwise convergence. Namely

$$f_n(x) \to f(x)$$

with $\mu(A) = 0$. □

2. Convergence almost uniformly implies convergence in measure.

Proof. As we write out the definition uniformly convergence like section (2), it suffices to consider the set

$$\{x \in D : |f_n(x) - f(x)| \geq \delta\} \subseteq A_\epsilon \quad \forall n \geq N$$

by taking the complement of the set $D \setminus A_\epsilon$. Hence we have

$$\mu(\{x \in D : |f_n(x) - f(x)| \geq \delta\}) \leq \mu(A_\epsilon) < \epsilon$$

It directly implies that

$$\lim_{n \to \infty} \mu(\{x \in D : |f_n(x) - f(x)| \geq \delta\}) = 0$$

□

3. L^p convergence implies convergence in measure.

To prove this fact, we first need an inequality below:

Lemma 1.5.1 (Chebyshev's Inequality) Suppose that

$$\int_D |f|^p d\mu < \infty$$

then we have an inequality

$$\mu(\{x \in D : |f(x)| > \epsilon\}) \leq \int_D |f|^p d\mu / \epsilon^p$$

Proof. Compute

$$\int_D |f|^p d\mu \geq \int_{\{x \in D : |f(x)| > \epsilon\}} |f|^p d\mu \geq \int_{\{x \in D : |f(x)| > \epsilon\}} \epsilon^p d\mu = \epsilon^p \mu(\{x \in D : |f(x)| > \epsilon\})$$

□

We will now prove that L^p convergence implies convergence in measure.

Proof. Apply the definition and the lemma above

$$\lim_{n\to\infty} \mu(\{x \in D : |f_n(x) - f(x)| \geq \epsilon\}) \leq \lim_{n\to\infty} \int_D |f_n - f|^p d\mu/\epsilon^p = 0$$

as desired. □

The following examples are some illustrations of how some converse implications do not apply.

Example 1.5.2 Convergence almost uniformly but not convergence in measure and hence not convergence almost uniformly.

Consider a sequence of functions

$$f_n = \chi_{[n,n+1)} \quad \text{for } n = 1, 2, \ldots$$

We have $f_n \to 0$ a.e. since the set $[n, n+1)$ has measure zero when n goes to infinity. However, we always have

$$\lim_{n\to\infty} \mu(\{x \in \mathbb{R} : |f_n| > 1/2\}) = 1$$

and it does not converge to zero in measure and hence not converge to zero almost uniformly.

Example 1.5.3 Convergence in measure but not convergence almost uniformly and hence not convergence almost uniformly.

Consider a sequence of functions

$$f_1 = \chi_{[0,1]}$$
$$f_2 = \chi_{[0,1/2]}$$
$$f_3 = \chi_{[1/2,1]}$$
$$f_4 = \chi_{[0,1/4]}$$
$$f_5 = \chi_{[1/4,2/4]}$$
$$f_6 = \chi_{[2/4,3/4]}$$
$$f_7 = \chi_{[3/4,1]}$$
$$f_8 = \ldots$$

It is easy to see that f_n diverges everywhere hence not convergence almost uniformly. However, as the measure of the interval goes to zero, we have

$$\lim_{n\to\infty} \mu(\{x \in [0,1] : |f_n(x) > \epsilon\}) = 0$$

and $f_n \xrightarrow{\mu} 0$ in measure.

Example 1.5.4 Consider a sequence of functions

$$f_n(x) = n\chi[0, 1/n]$$

It is easy to see that it converges to zero a.e., a.u., and in measure. However,

$$\int_{\mathbb{R}} |f_n| d\mu = 1$$

and it does not converge to zero in L^1 space.

Lecture 2:

6 The Borel-Cantelli Lemma and Its Corollary

Lemma 1.6.1 (The Borel-Cantelli Lemma) Suppose $\{E_n\}_{n=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : x \in E_n \text{ for infinitely many } n\} \\ &= \limsup_{n\to\infty} E_n \end{aligned}$$

Then the set E is measurable and $\mu(E) = 0$. (Note that it can also be extended to general measure space)

Proof. We will first prove that the set E is measurable, as we can write

$$E = \limsup_{k\to\infty}(E_k) = \bigcap_{n=1}^{\infty} \left(\bigcup_{k\geq n} E_k \right)$$

Since $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$, and countable union or intersection of measurable sets is measurable, then the set E as expressed above is also measurable.

To prove that the set has measure zero, define

$$F_n = \bigcup_{k\geq n} E_k$$

Then directly by construction, we can see that F_n is measurable(countable union of measurable sets), $F_{n+1} \subset F_n$, and $E = \bigcap_{n=1}^{\infty} F_n$. Also, since $m(F_1) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$ by countable sub-additivity, and we show $F_n \searrow E$ before, it implies that $m(E) = \lim_{n\to\infty} m(E_n)$.

Consider $m(E_k)$ as a sequence of number. Then if $\sum_{k=1}^{\infty} m(E_k) = L < \infty$, from the knowledge of series, we know that $\lim_{k\to\infty} m(E_k) = 0$ needs to be true. Therefore, by countable sub-additivity,

$$m(E) = \lim_{n\to\infty} m(F_n) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = \lim_{k\to\infty} m(E_k) = 0$$

Since $m(E) \geq 0$ by definition, $m(E) = 0$ is achieved as desired. □

The lemma has the following corollary as mentioned before in the section, to check the convergence a.e.:

Corollary 1.6.2 Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f : D \to \mathbb{R}$ and $\mathcal{A}$-measurable with $D \in \mathcal{A}$. Suppose that

1. $\exists a_n > 0$ such that $\lim_{n\to\infty} a_n = 0$
2. $$\sum_{n=1}^{\infty} \mu(\{x \in D : |f_n - f| \geq a_n\}) < \infty$$

then $f_n \to f$ a.e.

Proof. Given $\epsilon > 0$. Define

$$E_n^\epsilon = \{x \in D : |f_n - f| \geq \epsilon\}$$

From the condition (1), we have

$$\exists N \; \forall n > N \text{ s.t. } 0 < a_n < \epsilon$$

Now, for $n > N$, we have

$$F_n \coloneqq \{\{x \in D : |f_n - f| \geq a_n\}\} \supseteq E_n^\epsilon$$

By the Borel-Cantelli Lemma and the condition (2), we have

$$\mu(\limsup_{n\to\infty} F_n) = 0 \quad \Longrightarrow \quad \mu(\limsup_{n\to\infty} E_n^\epsilon) = 0$$

□

Although there is no direct relation between the convergence a.e. and convergence in measure, the following theorem shows us how they relates in a way.

Theorem 1.6.3 (Riesz Theorem) If $f_n \xrightarrow{\mu} f$, then $\exists f_{n_j} \to f$ *a.e.*

Proof. Since it converges in measure, we have

$$\mu(\{x \in D : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon \quad \text{for } n \geq N_\epsilon$$

Consider the following procedure

1. Pick n_1 s.t.
$$\mu(\{x \in D : |f_n(x) - f(x)| \geq 1/2^1\}) < 1/2^1$$

2. Pick n_2 s.t.
$$\mu(\{x \in D : |f_n(x) - f(x)| \geq 1/2^2\}) < 1/2^2$$

   $\vdots$

3. Pick n_j s.t.
$$\mu(\{x \in D : |f_n(x) - f(x)| \geq 1/2^j\}) < 1/2^j$$

Now as

$$\sum_j \mu(\{x \in D : |f_n(x) - f(x)| \geq 1/2^j\}) \leq 1 \quad \text{and} \quad 1/2^j \to 0$$

by the Corollary 1.6.2,

$$f_{n_j} \to f \ a.e.$$

□

7 Convergence in a Finite Complete Measure Space

Without specifying, the following conditions apply to this section: Given a *finite complete* measure space $(X, \mathcal{A}, \mu)$. With $D \in \mathcal{A}$, define a sequence of $\mathcal{A}$-measurable functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$.

Theorem 1.7.1 If $f_n \to f$ *a.e.*, then $f_n \xrightarrow{\mu} f$.

Proof. Given $\epsilon > 0$, define

$$E_n^\epsilon = \{x \in D : |f_n - f| \geq \epsilon\}$$

Since $f_n \to f$ *a.e.*, we have

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n^\epsilon\right) = 0$$

Define

$$F_k = \bigcup_{n \geq k} E_n^\epsilon$$

Then directly by construction, we can see that F_k is measurable (countable union of measurable sets) and $F_{k+1} \subseteq F_k$. Since the measure space is finite, we can write

$$0 = \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n^\epsilon\right) = \mu(\lim_{k\to\infty} F_k) = \lim_{k\to\infty} \mu(F_k)$$

Since $E_k^\epsilon \subseteq F_k$ and the measure space is complete, we then can conclude that

$$\lim_{k\to\infty} E_k^\epsilon \quad \forall \epsilon > 0 \quad \implies \quad f_n \xrightarrow{\mu} f$$

□

Theorem 1.7.2 (Egorov (Egoroff) Theorem) If $f_n \to f$ *a.e.*, then $f_n \to f$ *a.u.*

We will prove the generalization of the above theorem instead:

Theorem 1.7.2 (Generalized Egorov Theorem) Given a measure space $(X, \mathcal{A}, \mu)$. With $D \in \mathcal{A}$, define a sequence of $\mathcal{A}$-measurable functions $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$. Suppose that

1. $f_n \to f$ *a.e.*
2. $$\mu\left(\bigcup_{n=1}^{\infty} \left\{x \in X : |f_n - f| \geq \frac{1}{m}\right\}\right) < \infty \quad \forall m \in \mathbb{N}$$

then $f_n \to f$ *a.u.*

Proof. Given $\epsilon > 0$, we need to find $B \in \mathcal{A}$ and $\mu(B) < \epsilon$ s.t. $f_n \to f$ uniformly on B^c. Define

$$E = \bigcap_{k \geq 1} \bigcup_{n \geq k} \left\{x \in X : |f_n - f| \geq \frac{1}{m}\right\}$$

then $\mu(E) = 0$ for all $m \in \mathbb{N}$. Define

$$B_{k,m} = \bigcup_{n \geq k} \left\{x \in X : |f_n - f| \geq \frac{1}{m}\right\}$$

then $B_{k,m} \subseteq B_{k+1,m}$ and the second condition tells us that $\mu(B_{1,m}) < \infty$ for all m. Therefore

$$\mu(\lim_{k\to\infty} B_{k,m} = \mu\left(\bigcap_{k \geq 1} B_{k,m}\right) = \mu(E) = 0 \quad \implies \quad \lim_{k\to\infty} \mu(B_{k,m}) = 0$$

It implies that $\exists k_m$ s.t.

$$\mu(B_{k_m,m}) < \frac{\epsilon}{2^m}$$

Define

$$B \coloneqq \bigcup_{m\geq 1} B_{k_m,m}$$

then we have

$$\mu(B) \leq \sum_{m\geq 1} \mu(B_{k_m,m}) < \sum_{m\geq 1} \frac{\epsilon}{2^m} = \epsilon$$

and

$$B^c = \bigcap_{m\geq 1} B^c_{k_m,m}$$

Given $\delta > 0$, let $M \in \mathbb{N}$ such that $1/M < \delta$. If $x \in B^c$, from the intersection we must have $x \in B^c_{k_M,M}$, meaning that

$$|f_n - f| < \frac{1}{M} < \delta \quad \forall n > k_M$$

as the uniform convergence. □

Lecture 3

8 Other Convergence Theorems

Theorem 1.8.1 (Monotone Convergence Theorem) Given a measure space $(X, \mathcal{A}, \mu)$. Define a sequence of $\mathcal{A}$-measurable functions $f_n : X \to [0, \infty)$. Suppose that f_n is monotone increasing and

$$\lim_{n\to\infty} f_n = f \ a.e.$$

We then have

$$\int_X \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Proof. By the monotonicity, we have

$$\int_X \lim_{n\to\infty} f_n d\mu \geq \lim_{n\to\infty} \int_X f_n d\mu$$

Now it suffices to prove the other direction. To do this, we will approximate the function by simple function. Fix $0 \leq \phi(x) \leq f(x)$ for $\phi(x)$ simple function. Define

$$A_n = \{x \in X : f_n \geq \gamma\phi(x)\}$$

for $\gamma \in (0, 1)$. And that

$$A_{n+1} \supseteq A_n \quad \Longrightarrow \quad \mu(\lim A_n) = \lim \mu(A_n)$$

From the construction, we have

$$\int_X f_n \geq \int_{A_n} \gamma\phi(x)$$

It implies that

$$\lim_{n\to\infty} \int_X f_n \geq \lim_{n\to\infty} \int_{A_n} \gamma\phi(x) = \int_X \gamma\phi(x) = \gamma \int_X \phi(x)$$

As γ goes to 1, take the supreme of all the simple functions and we have

$$\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X \lim_{n\to\infty} f_n d\mu$$

□

Lemma 1.8.2 (Fatou's Lemma) Given a measure space $(X, \mathcal{A}, \mu)$. Define a sequence of $\mathcal{A}$-measurable functions $f_n : X \to [0, \infty)$. Then

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

Proof. Since

$$\liminf_{n\to\infty} f_n = \lim_{k\to\infty} \inf_{n\geq k} f_n$$

define

$$g(k) := \inf_{n\geq k} f_n$$

we have $g_k \leq g_{k+1}$. Therefore, by the M.C.T

$$\int \liminf f_n = \int \lim_{k\to\infty} g_k = \lim_{k\to\infty} \int g_k = \liminf_{k\to\infty} \int g_k \leq \liminf_{k\to\infty} \int f_k$$

as desired □

Theorem 1.8.3 (Dominated Convergence Theorem) Given a *complete* measure space $(X, \mathcal{A}, \mu)$. Define a sequence of $\mathcal{A}$-measurable functions $f_n : X \to \mathbb{R}$ and a $\mathcal{A}$-measurable $g : X \to [0, \infty)$ with $\int_X g d\mu < \infty$. Suppose that

1. $f_n \to f$ *a.e.*
2. $|f_n| \leq g$

then we have

1. $\int_X |f| d\mu < \infty$
2. The limit can be exchanged.
$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu$$
3. $f_n \to f$ in L^1 as
$$\limsup_{n\to\infty} \int_X |f_n - f| d\mu = 0$$

Proof. The first result is straightforward. By Fatou's Lemma, compute

$$\int |f| d\mu = \lim |f_n| d\mu \leq \lim \int |f_n| d\mu \leq \lim \int g d\mu < \infty$$

To prove the second result, since $|f_n| \leq g$, then both $f_n \leq g$ and $-g \leq f_n$ hold true. Apply the Fatou's Lemma twice to the functions $g - f_n$ and $g + f_n$ and obtain

$$\int \limsup f_n \geq \limsup \int f_n \quad \text{and} \quad \int \liminf f_n \leq \liminf \int f_n$$

As $f_n \to f$ *a.e.*, we have

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n \quad \Longrightarrow \quad \lim \int f_n = \int \lim f_n$$

To prove the third result, notice that $|f_n - f| \le 2g$. Apply the Fatou's Lemma and use the similar analogy of the first inequality in the proof of the second result, we have

$$0 = \int \limsup |f_n - f| \ge \limsup \int |f_n - f| \lim_{n\to\infty} \int |f_n - f| = 0$$

as desired. □

We need the following lemma to prove our next theorem.

Lemma 1.8.4 If every subsequence of $x_n \in \mathbb{R}$ has a subsequence that converges to a, then $x_n \to a$.

Proof. If not, there exists an $\epsilon > 0$, such that for all k, there exists an $n_k > k$ satisfying $|x_{n_k} - x| \ge \epsilon$ since if there is some k which doesn't have such n_k, then we can take it as N, so x_n converges to x. The subsequence x_{n_k} does not have any subsequence converging to x. □

Theorem 1.8.5 Suppose that $f_n \to f$ in μ and $|f_n| \le g$ and $\int g d\mu < \infty$, then $f_n \to f$ in L^1.

Proof. Now define

$$a_n = \int |f_n - f|$$

and

$$a_{n_k} = \int |f_{n_k} - f|$$

Since $f_n \overset{\mu}{\to} f$ in μ, then $\exists f_{n_j} \to f$ *a.e.* from Theorem 1.6.3 (Riesz Theorem). By D.C.T, we have that $a_{n_j} \to 0$ as $j \to \infty$. By the Lemma 1.8.4, we have $a_n \to 0$ as desired. □

Theorem 1.8.6 Given a measure space $(X, \mathcal{A}, \mu)$. Define a sequence of $\mathcal{A}$-measurable functions $f_n : X \to \mathbb{R}$ and a $\mathcal{A}$-measurable $g : X \to [0, \infty)$ with $\int_X g d\mu < \infty$. Suppose that $|f_n| \le g$. If $f_n \to f$ *a.e.*, then $f_n \to f$ *a.u.*

Proof. By Theorem 1.7.2 (Generalized Egorov Theorem), it suffices to check that

$$\mu\left(\bigcup_{n=1}^{\infty}\left\{x \in X : |f_n - f| \ge \frac{1}{m}\right\}\right) < \infty \quad \forall m \in \mathbb{N}$$

Since $|f_n| \le g$, then $|f_n - f| \le 2g$. Therefore,

$$\left\{x \in X : |f_n - f| \ge \frac{1}{m}\right\} \subseteq \left\{x \in X : g \ge \frac{1}{2m}\right\}$$

and

$$\mu\left(\left\{x \in X : |f_n - f| \ge \frac{1}{m}\right\}\right) \subseteq \mu\left(\left\{x \in X : g \ge \frac{1}{2m}\right\}\right)$$

Since $\int g d\mu < \infty$, the measure

$$\mu\left(\left\{x \in X : g \geq \frac{1}{2m}\right\}\right) < \infty \quad \implies \quad \mu\left(\left\{x \in X : |f_n - f| \geq \frac{1}{m}\right\}\right) < \infty$$

as desired. $\square$

Lecture 4

9 L^p space

We define the L^p space as the following. Let $(X, \mathcal{A}, \mu)$ be a measure space. We define the L^p norm as

$$\|f\|_{L^p(X)} \coloneqq \left(\int_X |f|^p \right)^{1/p}$$

and the L^p space as

$$L^p(X) \coloneqq \{f : X \to [-\infty, \infty\} \text{ s.t. } f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_{L^p(X)} < \infty$$

It is worth noticing that $L^p(X)$ consists of equivalence classes. Namely, for $f, g \in L^p(X)$, then

$$f = g \Leftrightarrow f = g \ a.e.$$

As a norm, the L^p norm satisfies the norm properties:

- $\|f\|_{L^p(X)} \Leftrightarrow f \equiv 0 \ a.e.$
- $\|cf\|_{L^p(X)} = |c| \, \|f\|_{L^p(X)}$
- $\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}$

The first two conditions are trivial, and the third will be proved at Inequality 1.9.2 The infinity L^p space is defined as the following:

$$L^\infty(X) \coloneqq \{f : X \to [-\infty, \infty\} \text{ s.t. } f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_{L^\infty(X)} < \infty$$

with the infinity norm

$$\|f\|_{L^\infty(X)} \coloneqq \inf\{M : \mu\{x \in X : |f| > M\} = 0\}$$

The properties of the infinity L^p space follow from the properties of L^p space.

Inequality 1.9.1 (Holder Inequality) If $1 \leq p, q \leq \infty$. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g : X \to [-\infty, \infty]$ be $\mathcal{A}$-measurable. If $f \in L^p(X)$, $g \in L^q$, and $1/p + 1/q = 1$, then

$$\left| \int_X fg d\mu \right| \leq \int_X |fg| d\mu \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}$$

and the values p and q are called the *Holder's dual* with one another.

Proof. The condition $1/p + 1/q = 1$ as we may consider in the following:

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(\lambda x)g(\lambda x)dx\right| &\leq \left(\int_{\mathbb{R}} |f(\lambda x)|^p dx\right)^{1/p} \left(\int_{\mathbb{R}} |g(\lambda x)|^q dx\right)^{1/q} \\
\left|\int_{\mathbb{R}} f(\zeta)g(\zeta)d\frac{\zeta}{\lambda}\right| &\leq \left(\int_{\mathbb{R}} |f(\zeta)|^p d\frac{\zeta}{\lambda}\right)^{1/p} \left(\int_{\mathbb{R}} |g(\zeta)|^q d\frac{\zeta}{\lambda}\right)^{1/q} \\
\frac{1}{\lambda}\left|\int_{\mathbb{R}} f(\zeta)g(\zeta)d\zeta\right| &\leq \frac{1}{\lambda^{1/p+1/q}} \left(\int_{\mathbb{R}} |f(\zeta)|^p d\zeta\right)^{1/p} \left(\int_{\mathbb{R}} |g(\zeta)|^q d\zeta\right)^{1/q}
\end{aligned}$$

then if we expected the inequality, we need to expect the equality

$$1/p + 1/q = 1$$

To retrieve the original proof, let's suppose that $p = \infty$ and $q = 1$ without loss of generality, then

$$\int |fg| \leq \|f\|_{L^\infty(X)} \int |g| = \|f\|_{L^\infty(X)} \|g\|_{L^1(X)}$$

with $|f| \leq \|f\|_{L^\infty(X)}$. Therefore, it suffices to consider that $1 < p, q < \infty$ and $\|f\|_{L^p(X)} \|g\|_{L^q(X)} \neq 0$. We then consider

$$\int |fg| \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)} \quad \Leftrightarrow \quad \int \frac{|f|}{\|f\|_{L^p(X)}} \frac{|g|}{\|g\|_{L^q(X)}} \leq 1$$

Define

$$\begin{cases} \text{F} := \dfrac{|f|}{\|f\|_{L^p(X)}} \\ \text{G} := \dfrac{|g|}{\|g\|_{L^q(X)}} \end{cases}$$

Note that $\|F\|_{L^p(X)} = \|G\|_{L^q(X)} = 1$ since they are normalized. Now it suffices to prove that $\int FG \leq 1$. From the convexity of the exponential function, consider the following

$$FG = e^{\ln FG} = e^{\ln F^p/p + \ln G^q/q} \leq \frac{e^{\ln F^p}}{p} + \frac{e^{\ln G^q}}{q} = \frac{F^p}{p} + \frac{G^q}{q}$$

Namely,

$$\int FG \leq \int \frac{F^p}{p} + \int \frac{G^q}{q} \leq \|F\|_{L^p(X)}/p + \|G\|_{L^q(X)}/q = \frac{1}{p} + \frac{1}{q} = 1$$

as desired. $\square$

We will now go back to prove the triangle inequality in $L^p(X)$.

Inequality 1.9.2 (Triangle inequality) The L^p norm satisfies the triangle inequality

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}$$

Proof. If $p = \infty$, the triangle is trivial. Suppose that $1 \leq p < \infty$, first notice that in the condition of the Holder's inequality, we have

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Longrightarrow \quad q = \frac{p}{p-1}$$

Now consider

$$\begin{aligned}
\|f+g\|_{L^p(X)}^p &= \int |f+g|^p d\mu \\
&= \int |f+g|^{p-1}|f+g| d\mu \\
&\leq \int |f+g|^{p-1}|f| d\mu + \int |f+g|^{p-1}|g| d\mu \\
&\text{by Holder's Inequality} \\
&\leq \left(\int |f+g|^{(p-1)q}\right)^{1/q} \|f\|_{L^p(X)} + \left(\int |f+g|^{(p-1)q}\right)^{1/q} \|g\|_{L^p(X)} \\
&= \left(\int |f+g|^{p}\right)^{1/q} \|f\|_{L^p(X)} + \left(\int |f+g|^{p}\right)^{1/q} \|g\|_{L^p(X)} \\
&= \|f+g\|_{L^p}^{p/q} \cdot (\|f\|_{L^p} + \|g\|_{L^p})
\end{aligned}$$

It implies that

$$\|f+g\|_{L^p}^{p(1-1/q)} \leq \|f\|_{L^p} + \|g\|_{L^p} \quad \Longrightarrow \quad \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

as desired. □

After we proved that the L^p norm is indeed a norm, we will now show that L^p space is a Banach space. We first establish the following lemma

Lemma 1.9.3 A normed linear Space X is a Banach Space if and only if every absolutely convergent series is convergent.

Scratch proof. the implication can be proved by the argument that the partial sum of an absolutely convergent series is a Cauchy sequence, hence converges. The converse can be proved by the fact that every Cauchy sequence is convergent from the argument that we can construct an absolutely convergent subsequence from the Cauchy sequence.

Theorem 1.9.4 L^p space is complete.

Proof. From the Lemma 1.9.3, it suffices to show that the absolute convergence implies the L^p convergence. Let a sequence of function $f_k \in L^p$ and is absolutely convergent

$$\sum_{k=1}^{\infty} \|f_k\|_{L^p} = M < \infty$$

Define

$$G_n = \sum_{k=1}^{n} |f_k| \quad \text{and} \quad G = \sum_{k=1}^{\infty} |f_k|$$

with $\lim G_n \to G$ *a.e.* Directly from the definition

$$\|G_n\|_{L^p} \leq \sum_{k=1}^{n} \|f_k\|_{L^p} \leq M \quad \forall n$$

and by the Monotone Convergence Theorem, we have

$$\int G^p = \int \lim G_n^p = \lim \int G_n^p \leq M^p$$

Hence $G \in L^p$ and in particular $G(x) < \infty$ *a.e.*. We now define

$$F := \sum_{k=1}^{\infty} f_k$$

which converges a.e. from the fact that $G < \infty$ *a.e.*. Also from the definition, we have

$$F \leq G \quad \Longrightarrow \quad F \in L^p$$

Moreover,

$$\left| F - \sum_{k=1}^{n} f_k \right|^p \leq (2G)^p \in L^1$$

By the Dominated Convergence Theorem, we obtain

$$\left\| F - \sum_{k=1}^{n} f_k \right\|_p^p = \int \left| F - \sum_{k=1}^{n} f_k \right|^p \to 0$$

Thus the series $\sum_{k=1}^{\infty} f_k$ converges in the L^p norm. □

After checking the properties of the L^p space, it is important to know the relations between the L^p spaces.

Theorem 1.9.5 If $\mu(X) < \infty$, and $1 \leq p < q \leq \infty$, then $L^p \supseteq L^q$.

Proof. Define $\gamma \in [1, \infty]$ by

$$\frac{1}{q} + \frac{1}{\gamma} = \frac{1}{p} \quad \Longrightarrow \quad \frac{p}{q} + \frac{p}{\gamma} = 1$$

For any $f \in L^q$, by Holder's inequality, we have

$$\|f\|_p^p = \int_X |f|^p d\mu \leq \|f^p\|_{q/p} \cdot \|1\|_{\gamma/p} = \left(\int_X f^{p \cdot q/p} d\mu \right)^{p/q} \cdot \mu(x)^{p/\gamma} = \|f\|_p^p \cdot \mu(x)^{p/\gamma}$$

Therefore,

$$\|f\|_p^p \leq \|f\|_q \cdot \mu(x)^{1/\gamma} < \infty$$

as desired. □

In general, we have $f \in L^p(X) \nRightarrow f \in L^q(X)$ for different values $p \neq q$. Here are some examples,

- For $X = (0,1)$
  - $1/\sqrt{x} \in L^1(X)$, but $1/\sqrt{x} \notin L^2(X)$.
  - $\ln 1/x \in L^p(X)$, but $\ln 1/x \notin L^\infty(X)$.
  - $\ln x \in L^p(X)$ for all $p \in [1,\infty)$, but $\ln x \notin L^\infty(X)$.
- For $X = (1,\infty)$, $1/x \in L^2(X)$, but $1/x \notin L^1(X)$.

Although there is no convergence of the L^p spaces, there is a convergence of the L^p norm.

Theorem 1.9.6 For finite measure space, the limsup of the L^p norm converges to the L^∞ norm. Namely, for $\mu(X) < \infty$,

$$\|f\|_{L^\infty} = \limsup_{p\to\infty} \|f\|_{L^p}$$

Proof. For one direction, we have

$$\left(\int |f|^p d\mu\right)^{1/p} \leq \|f\|_{L^\infty} \mu(X)^{1/p}$$

which implies

$$\limsup_{p\to\infty} \left(\int |f|^p d\mu\right)^{1/p} \leq \limsup_{p\to\infty} \|f\|_{L^\infty} \mu(X)^{1/p} = \|f\|_{L^\infty}$$

For another direction, fix $c < \|f\|_{L^\infty}$, then

$$\mu\{x \in X : |f| > c\} \neq 0$$

Now it suffices to show that

$$\limsup_{p\to\infty} \|f\|_{L^\infty} \geq c$$

Consider

$$\left(\int_X |f|^p d\mu\right)^{1/p} \geq \left(\int_{|f|>c} |f|^p d\mu\right)^{1/p} \geq c\mu(|f| > c)^{1/p}$$

and if we take the limsup from both sides, we obtain the inequality $\limsup_{p\to\infty} \|f\|_{L^\infty} \geq c$. From both sides of the inequalities, the proof is complete. $\square$

Lecture 5

10 Interpolation Theorems

We will first prove the following theorem

Theorem 1.10.1 If $1 \leq p < r < q \leq \infty$, then $L^p \cap L^q \subset L^r$ and

$$\|f\|_r \leq \|f\|_p^\lambda \|f\|_q^{1-\lambda}$$

where $\lambda \in (0,1)$ is defined by

$$\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$$

Proof. If $q = \infty$, then $\lambda = p/r$, and the proof is trivial. Now consider $q < \infty$, and note that

$$\frac{1}{p(\lambda r)^{-1}} + \frac{1}{q[(1-\lambda)r]^{-1}} = 1$$

Using the Holder's inequality,

$$\begin{aligned}
\int |f|^r &= \int |f|^{\lambda r} |f|^{(1-\lambda)r} \\
&\leq \| \| |f|^{\lambda r}{}_{p/(\lambda r)} \cdot \left\| |f|^{(1-\lambda)r} \right\|_{q/[(1-\lambda)r]} \\
&= \left(\int |f|^p \right)^{\lambda r/p} \left(\int |f|^q \right)^{[(1-\lambda)r]/q} \\
&= \|f\|_p^{\lambda r} \|f\|_q^{(1-\lambda)r}
\end{aligned}$$

Taking the r-th roots, we are done. □

This theorem is actually a special case of the following theorem.

Theorem 1.10.2 (Riesz-Thorin Interpolation Theorem) Suppose that $(X_1, \mathcal{A}_1, \mu_X)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are measure spaces and $p_1, p_2, q_1, q_2 \in \infty$. For $0 < t < 1$, define p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_1} + \frac{t}{p_2} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_1} + \frac{t}{q_2}$$

If T is a linear map such that

$$T : L^{p_1}(X_1) + L^{p_2}(X_1) \to L^{q_1}(X_2) + L^{q_2}(X_2)$$

such that the following two kinds of mapping are bounded, say

- For $f \in L^{p_1}(X_1)$
$$\|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$$
- For $f \in L^{p_2}(X_1)$
$$\|Tf\|_{q_2} \leq M_2 \|f\|_{p_2}$$

then for $f \in L^{p_t}(X_1)$ and $0 < t < 1$, we have

$$\|Tf\|_{q_t} \leq M_1^{1-t} M_2^t \|f\|_{p^t}$$

We haven't got the tools to prove this theorem, and we will come back later. It is presented here because its close relationship with the theorem before.

Theorem 1.10.3 If $1 \leq p < r < q \leq \infty$, then $L^r \subset L^p + L^q$. That is, each $f \in L^r$ is the sum of a function in L^p and a function in L^q.

Proof. If $f \in L^r$, define $E \coloneqq \{x : |f(x)| > 1\}$ and set

$$g = f\chi_E \quad \text{and} \quad h = f\chi_{E^c}$$

Then we have

$$|g|^p = |f|^p \chi_E \leq |f|^r \chi_E \quad \implies \quad g \in L^p$$

and

$$|h|^q = |f|^q \chi_{E^c} \leq |f|^r \chi_{E^c} \quad \implies \quad h \in L^q$$

as desired. ☐

To setup the general setting of the interpolation theorem, we will introduce a new function space as the following

Definition 1.10.4 If f is a measurable function on $(X, \mathcal{A}, \mu)$, we define its *distribution function* m_f : $(0, \infty) \to [0, \infty]$ by

$$m_f(\lambda) = \mu(\{x : |f| > \lambda\}$$

Definition 1.10.5 Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, $1 \leq p < \infty$, we say that a function is in the *weak L^p space* as $f \in L^{p,weak}(X)$ if $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable and

$$\mu(\{x \in X : |f| > \lambda\}) \leq \frac{C^p}{\lambda^p}$$

for all $\lambda > 0$ and some constant c.

From the Chebyshev's inequality, it is easy to see that $f \in L^p \subseteq L^{p,weak}(X)$. This space is equipped with a quasi-norm

$$\|f\|_{L^{p,weak}(X)} \coloneqq \sup_{\lambda>0} \lambda m_f^{1/p}(\lambda)$$

Note that a quasi-norm is not a norm, namely, it does not satisfy the triangle inequality.

The following theorem illustrates the L^p norm and the distribution function.

Theorem 1.10.6 (Layer Cake Representation) Let f be measurable, $1 \leq p < \infty$, we have

$$\int_X |f|^p d\mu = p \int_0^\infty t^{p-1} m_f(t) dt$$

In particular, we have

$$\int_X |f| d\mu = \int_0^\infty m_f(t) dt$$

Proof. Compute

$$\begin{aligned}
\int_X |f|^p d\mu &= \int_X \left(\int_0^{|f(x)|} pt^{p-1} dt \right) d\mu \\
&= \int_X \left(\int_0^\infty pt^{p-1} \chi_{[0,|f(x)|)}(t) dt \right) d\mu \\
&= \int_0^\infty \left(\int_X pt^{p-1} \chi_{[0,|f(x)|)}(t) d\mu \right) dt \qquad \text{by Tonelii's theorem} \\
&= \int_0^\infty pt^{p-1} \left(\int_X \chi_{\{x:|f(x)|>t\}}(x) d\mu \right) dt \\
&= p \int_0^\infty t^{p-1} \mu(x : |f(x)| > t) dt
\end{aligned}$$

as desired. □

Theorem 1.10.7 (Marcin Kiewicz Interpolation, 1939) Define an operator T as

$$T : L^p(X) + L^q(X) \to L^{p,weak}(X) + L^{q,weak}(X)$$

and let it be subadditive as

$$|T(f+g)| \leq |Tf| + |Tg|$$

If the following two mappings are bounded

- $T : L^p \to L^{p,weak}$
- $T : L^q \to L^{q,weak}$

that is,

$$\mu(\{x \in X : T|f| > \lambda\}) \leq \frac{f_{L^p}^p}{\lambda^p}, \frac{f_{L^q}^q}{\lambda^p q}$$

then the mapping $T : L^r \to L^r$ is also bounded for $p < r < q$.

Proof. We want to show that

$$\|Tf\|_r^r \leq C\,\|f\|_r^r$$

for some constant C independently. By the Layer Cake Representation, we have

$$\int |Tf|^r d\mu = r\int_0^\infty \lambda^{p-1}\mu\{x \in X : |Tf| > \lambda\}d\lambda$$

We can rewrite

$$f = f_1 + f_2 = f\chi_{\{|f|\leq\lambda\}} + f\chi_{\{|f|>\lambda\}}$$

By subadditive, we have

$$|Tf| = |T(f_1 + f_2)| \leq |Tf_1| + |Tf_2|$$

From the two results above, we have

$$\{x \in X : |Tf| > \lambda\} \subseteq \{x \in X : |Tf_1| > \lambda/2\} \cup \{x \in X : |Tf_2| > \lambda/2\}$$

and

$$\begin{aligned}\mu\{x \in X : |Tf| > \lambda\} &\leq \mu\{x \in X : |Tf_1| > \lambda/2\} + \mu\{x \in X : |Tf_2| > \lambda/2\}\\ &\leq \frac{\|f_1\|_q^q}{\lambda^q} + \frac{\|f_2\|_p^p}{\lambda^p}\end{aligned}$$

By Tonelli's Theorem, compute separately

$$\begin{aligned}r\int_0^\infty \lambda^{r-1}\frac{\|f_1\|_q^q}{\lambda^q}d\lambda &= r\int_0^\infty \lambda^{r-q-1}\int_{|f|<\lambda}|f|^q d\mu d\lambda\\ &= \int |f|^q \int_{|f|}^\infty r\lambda^{r-q-1}d\lambda d\mu\\ &= \int |f|^q \cdot \left[\frac{r}{r-q}\lambda^{r-q}\right]\Big|_{|f|}^\infty d\mu\\ &= \int |f|^q \frac{r}{r-q}|f|^{r-q}d\mu\\ &= \frac{r}{r-q}\int |f|^r d\mu\end{aligned}$$

Similarly, we have

$$\begin{aligned}
r\int_0^\infty \lambda^{r-1}\frac{\|f_1\|_p^p}{\lambda^p}d\lambda &= r\int_0^\infty \lambda^{r-p-1}\int_{|f|>\lambda}|f|^p d\mu d\lambda \\
&= \int |f|^p \int_0^{|f|} r\lambda^{r-p-1}d\lambda d\mu \\
&= \int |f|^p \cdot \left[\frac{r}{r-p}\lambda^{r-p}\right]\Big|_0^{|f|} d\mu \\
&= \int |f|^p \frac{r}{r-p}|f|^{r-p}d\mu \\
&= \frac{r}{r-p}\int |f|^r d\mu
\end{aligned}$$

Therefore,

$$\|Tf\|_r^r \leq \left(\frac{r}{q-r}+\frac{r}{r-p}\right)\|f\|_r^r$$

as desired. □